Numerical Experiments on The Capacity of Quantum Channel with Entangled Input States

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Abstract

The problem whether entangled input states can increase the capacity of quantum channel is investigated. We apply one of the quantum Arimoto-Blahut type algorithms to this problem and see that the results suggest the negative answer[1].

1 Introduction

The coding theorem for quantum channel was proved in recent publications [7, 12] combined with a pioneer work [6]. This theorem gives the formula of the capacity of quantum channel with product input states. However, whether entangled input states can increase the capacity of quantum channel is open. Moreover, this question can be regarded as a special case of a more general problem whether the capacity of product quantum channel exhibits additivity. In the present study, the latter problem is examined numerically.

2 Quantum channel and its capacity

2.1 Quantum channel with product input states

In the beginning, we will review the standard notion of quantum channel with product input states.

Let $\mathcal{H}$ be a Hilbert space which corresponds to a quantum system. A quantum state is represented by a density operator on $\mathcal{H}$, i.e. non-negative operator with unit trace. We denote by $\mathcal{S}(\mathcal{H})$ the totality of density operators on $\mathcal{H}$.

Letting $\mathcal{H}_1$ and $\mathcal{H}_2$ be input and output system, a quantum channel is described by a completely positive trace preserving linear map [13]

$$\Gamma \ : \ \mathcal{T}(\mathcal{H}_1) \to \mathcal{T}(\mathcal{H}_2)$$

where $\mathcal{T}(\mathcal{H}_1)$ and $\mathcal{T}(\mathcal{H}_2)$ are the totalities of the trace class operators on $\mathcal{H}_1$ and $\mathcal{H}_2$.

[1]This work was partly presented at the second QIT [10] and the 22nd SITA [11].

A quantum communication system in which a quantum channel Γ is used n times is described as follows. A message set $\mathcal{M}_n := \{1, 2, \cdots, M_n\}$ denotes the totality of the messages which are to be transmitted. Each message $k \in \mathcal{M}_n$ is encoded to a codeword which is a product state in the form $\rho^{(n)}(k) := \rho_1(k) \otimes \cdots \otimes \rho_n(k)$ on $\mathcal{H}_1^{\otimes n}$, where $\mathcal{H}_1^{\otimes n}$ denotes a tensor product Hilbert space $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_1$. The sender transmits the codeword by multiple use of a quantum channel Γ. Then the received state is also a product state $\Gamma^{\otimes n}(\rho^{(n)}(k)) = \Gamma(\rho_1(k)) \otimes \cdots \otimes \Gamma(\rho_n(k))$ on $\mathcal{H}_2^{\otimes n}$. Here $\Gamma^{\otimes n}$ denotes the n-fold tensor product channel $\Gamma \otimes \cdots \otimes \Gamma$ acting on $\mathcal{T}(\mathcal{H}_1^{\otimes n})$. The receiver estimates which codeword has been actually transmitted by performing a $\mathcal{M}_n$-valued measurement. Mathematically, this measurement is described by a positive operator valued measure (POVM) $X^{(n)} = \{X_1^{(n)}, \cdots, X_{M_n}^{(n)}\}$ on $\mathcal{H}_2^{\otimes n}$, i.e. $X_k^{(n)} \geq 0\,(k = 1, \cdots, M_n)$ and $\sum_{k=1}^{M_n} X_k^{(n)} = I$. We denote by Φ_n the *coding system* which consists of codewords $\{\rho^{(n)}(k)\}_{k=1}^{M_n}$ and a measurement $X^{(n)}$. The error probability averaged over all codewords with a fixed coding system Φ_n is given by

$$P_{er}(\Phi_n, \Gamma) = 1 - \frac{1}{M_n} \sum_{k=1}^{M_n} \mathrm{Tr}\,[\Gamma^{\otimes n}(\rho^{(n)}(k)) X_k^{(n)}]. \tag{1}$$

The quantity $R_n = \log M_n / n$ is called the *rate* for the coding system Φ_n. The (operational) capacity of the quantum channel Γ with product input states is defined as

$$C(\Gamma) := \sup\{\lim_{n\to\infty} \frac{1}{n} \log M_n\ ;\ \lim_{n\to\infty} P_{er}(\Phi_n, \Gamma) = 0\}. \tag{2}$$

Next we will explain the quantum mutual information. Let

$$\begin{aligned} \Pi_n := \{\pi = (\lambda_1, \cdots, \lambda_n\,;\, \sigma_1, \cdots, \sigma_n)\ ; \\ 0 \leq \lambda_i \in \mathbf{R},\ \sum_{i=1}^n \lambda_i = 1,\ \sigma_i \in \mathcal{S}(\mathcal{H}_1)\}, \end{aligned}$$

$$\Pi := \bigcup_n^\infty \Pi_n.$$

Note that $\pi \in \Pi$ is a discrete probability distribution on $\mathcal{S}(\mathcal{H}_1)$ assigning probability λ_i to the state σ_i. The quantum mutual information for π and Γ is defined as

$$\mathrm{I}(\pi; \Gamma) := \sum_j \lambda_j \mathrm{D}(\Gamma(\sigma_j) \| \Gamma(\rho)), \tag{3}$$

where $\rho := \sum_j \lambda_j \sigma_j$ is a convex combination of the states in π and $\mathrm{D}(\rho\|\sigma) := \mathrm{Tr}\,[\rho(\log \rho - \log \sigma)]$ is the quantum relative entropy.

The quantum channel coding theorem [6, 7, 12] states that

$$C(\Gamma) = \sup_{\pi \in \Pi} \mathrm{I}(\pi; \Gamma). \tag{4}$$

In addition, supremization is reduced to maximization on certain finite-dimensional compact set [4]. That is,

$$C(\Gamma) = \max_{\pi \in \Pi_n} \mathrm{I}(\pi) = \max_{\pi \in \Pi_n^e} \mathrm{I}(\pi)$$

where $n = \dim \Gamma(\mathcal{S}(\mathcal{H}_1)) + 1$, $\Pi_n^e := \{(\lambda_i; \sigma_i) \in \Pi_n\ ;\ \sigma_i \in \partial_e \mathcal{S}(\mathcal{H}_1),\ i = 1, \cdots, n\}$. Here $\partial_e \mathcal{S}(\mathcal{H}_1)$ is the totality of extreme points (pure states) of $\mathcal{S}(\mathcal{H}_1)$.

2.2 Quantum channel with entangled input states

Some states on a tensor product Hilbert space cannot be represented as product states or their convex combination. These states are called *entangled states*. So far we have restricted ourselves to product input states. Now let us consider a quantum channel with entangled input states.

The (operational) capacity of quantum channel with entangled input states $\tilde{C}(\Gamma)$ is defined in the same way as (2) except that arbitrary states on $\mathcal{H}_2^{\otimes n}$ can be codewords. It is obvious by definition that

$$\tilde{C}(\Gamma) \geq C(\Gamma). \tag{5}$$

However, it is not clear whether much stronger statement

$$\tilde{C}(\Gamma) = C(\Gamma) \tag{6}$$

always holds.

2.3 Product quantum channel

Let $\Gamma^{(i)} : \mathcal{T}(\mathcal{H}_1^{(1)}) \to \mathcal{T}(\mathcal{H}_2^{(2)})$, $(i = 1, 2)$ be quanutm channels and let $\Gamma^{(1)} \otimes \Gamma^{(2)} : \mathcal{T}(\mathcal{H}_1^{(1)} \otimes \mathcal{H}_1^{(2)}) \to \mathcal{T}(\mathcal{H}_2^{(1)} \otimes \mathcal{H}_2^{(2)})$ be their product channel. The capacity $C(\Gamma^{(1)} \otimes \Gamma^{(2)})$ is defined as (2) by replacing Γ with $\Gamma^{(1)} \otimes \Gamma^{(2)}$ in which each input state is written in the form $\rho^{(n)}(k) = \rho_1(k) \otimes \cdots \otimes \rho_n(k)$, where $\rho_i(k)$ $(i = 1, \cdots, n)$ are arbitrary states on $\mathcal{H}_1^{(1)} \otimes \mathcal{H}_1^{(2)}$. Then it is easy to see that the superadditivity

$$C(\Gamma^{(1)} \otimes \Gamma^{(2)}) \geq C(\Gamma^{(1)}) + C(\Gamma^{(2)}) \tag{7}$$

holds. However, the question whether we have always the equality is open. In fact, this question includes the previous mentioned question. This is because the following theorem holds. (The proof is given in Appendix A.)

Theorem.

$$\tilde{C}(\Gamma) = \lim_{N \to \infty} \frac{C(\Gamma^{\otimes N})}{N} = \sup_N \frac{C(\Gamma^{\otimes N})}{N} \tag{8}$$

holds. Here $C(\Gamma^{\otimes N})$ is defined as (2) by replacing Γ with $\Gamma^{\otimes N}$ in which each input state is written in the form $\rho^{(n)}(k) = \rho_1(k) \otimes \cdots \otimes \rho_n(k)$, where $\rho_i(k)$ $(i = 1, \cdots, n)$ are arbitrary states on $\mathcal{H}_1^{\otimes N}$.

Therefore, if the additivity of (7) always holds, entanglement of input states cannot increase the capacity of quantum channel.

3 Numerical experiments

3.1 Quantum version of Arimoto-Blahut algorithm

The Arimoto-Blahut algorithm is known for computing the capacity of classical channel [1, 2]. Recently, one of the authors proposed two algorithms of this type for computing the capacity of quantum channel [9, 10]. We use one of these. It is called the *boundary* *algorithm* since its recursion works on $\partial_e \mathcal{S}(\mathcal{H}_1)$. The outline of the theoretical basis is as follows. Let us introduce a two-variable extension of $\mathrm{I}(\pi;\Gamma)$:

$$J(\pi, \pi') := -\mathrm{D}(\lambda \| \lambda') + \sum_{i=1}^{n} \lambda_i \mathrm{Tr}\,[\Gamma(\sigma_i)\Phi(\sigma'_i, \rho')], \tag{9}$$

where

$$\pi = (\lambda_i; \sigma_i),\ \pi' = (\lambda'_i; \sigma'_i) \in \Pi_n,$$
$$\mathrm{D}(\lambda\|\lambda') := \sum_{i=1}^{n} \lambda_i \log \frac{\lambda_i}{\lambda'_i},\ \rho' := \sum_{i=1}^{n} \lambda'_i \sigma'_i,$$
$$\Phi(\sigma'_i, \rho') := \log(\Gamma(\sigma'_i)) - \log(\Gamma(\rho')).$$

Then it holds that

$$\mathrm{I}(\pi;\Gamma) = J(\pi,\pi) = \max_{\pi'} J(\pi, \pi'). \tag{10}$$

We can compute $\hat{\pi} = (\hat{\lambda}_i,\, \hat{\sigma}_i) := \mathrm{argmax}_{\pi}\, J(\pi,\pi')$ by the following equations.

$$\hat{\sigma}_i = \operatorname*{argmax}_{\sigma \in \mathcal{S}(\mathcal{H}_1)} \mathrm{Tr}\,[\Gamma(\sigma)\Phi(\sigma'_i, \rho')],$$

$$\hat{\lambda}_i = \lambda'_i \exp(\mathrm{Tr}\,[\Gamma(\hat{\sigma}_i)\Phi(\sigma'_i,\rho')])/\hat{Z},$$

where $\hat{Z}$ is the normalizing constant:

$$\hat{Z} := \sum_{i=1}^{n} \lambda'_i \exp(\mathrm{Tr}\,[\Gamma(\hat{\sigma}_i)\Phi(\sigma'_i,\rho')]).$$

Note that, since $\mathrm{Tr}\,[\Gamma(\sigma)\Phi(\sigma'_i,\rho')]$ is linear in σ, we can always choose $\hat{\sigma}_i$ to be an extreme point of $\mathcal{S}(\mathcal{H}_1)$, i.e. a pure state $|\psi_i\rangle\langle\psi_i|$, where $|\psi_i\rangle$ is a normalized eigenvector of $\Gamma^*(\Phi(\sigma'_i,\rho'))$ corresponding to the maximum eigenvalue. Here $\Gamma^* : \mathcal{B}(\mathcal{H}_2) \to \mathcal{B}(\mathcal{H}_1)$ denotes the dual map of Γ defined by $\mathrm{Tr}\,[\Gamma(X)Y] = \mathrm{Tr}\,[X\Gamma^*(Y)]$ for $\forall X \in \mathcal{T}(\mathcal{H}_1)$ and $\forall Y \in \mathcal{B}(\mathcal{H}_2)$, where $\mathcal{B}(\mathcal{H}_i)$ $(i = 1, 2)$ are the totalities of the bounded operators on $\mathcal{H}_i$.

For given n and arbitrary initial element $\pi^{(1)} \in \Pi_n$, let the sequence $\{\pi^{(k)}\}_{k=1}^{\infty}$ be defined by

$$\pi^{(k+1)} := \operatorname*{argmax}_{\pi} J(\pi, \pi^{(k)}). \tag{11}$$

Note that the sequence $\{\mathrm{I}(\pi^{(k)};\Gamma)\}_{k=1}^{\infty}$ is monotonous, since

$$\mathrm{I}(\pi^{(k)};\Gamma) \le J(\pi^{(k+1)}, \pi^{(k)}) \le \mathrm{I}(\pi^{(k+1)};\Gamma) \tag{12}$$

holds. Therefore we can efficiently compute the limit value $\lim_{k\to\infty} \mathrm{I}(\pi^{(k)};\Gamma)$. Unfortunately, it is not necessarily the quantum channel capacity since the quantum version of Arimoto-Blahut algorithm does not assure the global maximum. Thus we make several convergent sequences and adopt the maximum limit value as an estimate of the capacity. We judge that a sequence reaches the limit value when ten successive numerical values are the same to six places of decimals.

3.2 Results

We carry out numerical experiments to investigate the additivity of the capacity of product quantum channel in $\mathcal{H}_1 = \mathcal{H}_2 = \mathbf{C}^2$ and $\mathcal{H}_1 = \mathcal{H}_2 = \mathbf{C}^3$. Table 1 and 2 show the examples of quantum channels investigated. The quantum channels acting on $\mathcal{T}(\mathbf{C}^2)$ are denoted by the form (A, b) (see Appendix C) and those acting on $\mathcal{T}(\mathbf{C}^3)$ are denoted by their generators (see Appendix B). The results of numerical experiments are shown in Table 3. These results suggest that the additivity of product quantum channel capacity always holds. Figure 1 illustrates an example of a change in quantum mutual information $\mathrm{I}(\pi^{(k)}, \Gamma^{(1)} \otimes \Gamma^{(2)})$ starting from some entangled states in $\mathcal{S}(\mathcal{H}_1^{(1)} \otimes \mathcal{H}_1^{(2)})$. In addition, we measure the entanglement of the states in $\pi^{(k)} = (\lambda_i^{(k)}; \sigma_i^{(k)})$ by

$$\mathrm{Ent}(\pi^{(k)}) := \sum_i \lambda_i^{(k)} \mathrm{D}(\sigma_i^{(k)} \| \sigma_{i1}^{(k)} \otimes \sigma_{i2}^{(k)}), \tag{13}$$

where $\sigma_{i1}^{(k)}$ and $\sigma_{i2}^{(k)}$ are the marginal states of $\sigma_i^{(k)}$ defined by partial trace. Figure 2 shows how the states get disentangled through the recursion.

Moreover, the present experiments suggest that the probability distribution $\pi^* := \operatorname*{argmax}_\pi \mathrm{I}(\pi; \Gamma^{(1)} \otimes \Gamma^{(2)})$ is the product probability distribution of $\pi_1^* = (\lambda_{i1}^*; \sigma_{i1}^*)$ and $\pi_2^* = (\lambda_{j2}^*; \sigma_{j2}^*)$, assigning probability $\lambda_{i1}^* \lambda_{j2}^*$ to the state $\sigma_{i1}^* \otimes \sigma_{j2}^*$, where $\pi_1^* := \operatorname*{argmax}_\pi \mathrm{I}(\pi; \Gamma^{(1)})$ and $\pi_2^* := \operatorname*{argmax}_\pi \mathrm{I}(\pi; \Gamma^{(2)})$.

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Appendix A: Proof of the theorem

Since $C(\Gamma^{\otimes N+M}) \geq C(\Gamma^{\otimes N}) + C(\Gamma^{\otimes M})$ holds, $\lim_{N\to\infty} \dfrac{C(\Gamma^{\otimes N})}{N}$ exists and is proved to be $\sup_N \dfrac{C(\Gamma^{\otimes N})}{N}$. Let $\{\Phi_N\}_N$ be a sequence of coding system whose codewords are arbitrary states on $\mathcal{H}_1^{\otimes N}$ and assume that $\lim_{N\to\infty} P_{er}(\Phi_N, \Gamma) = 0$. Let Y be the classical random variable describing the transmitted message and be taken with the input distribution assigning equal probability $1/\mathcal{M}_N$ to each message, and let $\hat{Y}$ be the classical random variable describing the output of the product channel $\Gamma^{\otimes N}$ under the measurement $X^{(N)}$. The Fano inequality (see e.g. [3]) implies

$$1 + P_{er}(\Phi_N, \Gamma) \log \mathcal{M}_N \geq \log \mathcal{M}_N - \mathrm{I}(Y; \hat{Y}), \tag{14}$$

where $\mathrm{I}(Y; \hat{Y})$ is the classical mutual information. This leads to

$$(1 - P_{er}(\Phi_N, \Gamma)) \frac{1}{N} \log \mathcal{M}_N \leq \frac{1}{N} + \frac{1}{N} \mathrm{I}(Y; \hat{Y}). \tag{15}$$

In addition, we have

$$\begin{aligned}
\mathrm{I}(Y;\hat{Y}) &= \sum_{i=1}^{\mathcal{M}_N} \frac{1}{\mathcal{M}_N} \mathrm{D}(P_{\hat{Y}|Y}(\cdot|i)\|P_{\hat{Y}}) \\
&= \sum_{i=1}^{\mathcal{M}_N} \frac{1}{\mathcal{M}_N} \mathrm{D}_{X^{(N)}}(\Gamma^{\otimes N}(\sigma^{(N)}(i))\|\Gamma^{\otimes N}(\rho^{(N)})) \\
&\le \sum_{i=1}^{\mathcal{M}_N} \frac{1}{\mathcal{M}_N} \mathrm{D}(\Gamma^{\otimes N}(\sigma^{(N)}(i))\|\Gamma^{\otimes N}(\rho^{(N)})) \\
&\le \max_{\pi} \mathrm{I}(\pi;\Gamma^{\otimes N}) \\
&= C(\Gamma^{\otimes N}), \qquad (16)
\end{aligned}$$

where $\rho^{(N)} = \sum_i \frac{1}{\mathcal{M}_N}\sigma^{(N)}(i)$. Here we use the monotonicity of relative entropy. Substituting (16) into (15), letting $\mathrm{N} \to \infty$ and taking supremum with respect to $\{\Phi_N\}_N$, we come to the inequality $\tilde{C}(\Gamma) \le \lim_{N\to\infty} \dfrac{C(\Gamma^{\otimes N})}{N}$. Conversely, since $C(\Gamma^{\otimes N})$ is the supremum of the limit values of the rates of asymptotically error-free coding systems whose codewords are restricted to product states of the form $\rho_1 \otimes \cdots \otimes \rho_n \in \mathcal{S}(\mathcal{H}_1^{\otimes Nn})$, where ρ_i $(i = 1, \cdots, n)$ are arbitrary states on $\mathcal{H}_1^{\otimes N}$, it cannot be greater than $N\tilde{C}(\Gamma)$. Hence we have $\tilde{C}(\Gamma) \ge \lim_{N\to\infty} \dfrac{C(\Gamma^{\otimes N})}{N}$.

Appendix B: Operator-sum representation

Arbitrary completely positve trace preserving linear map $\Gamma : \mathcal{T}(\mathcal{H}_1) \to \mathcal{T}(\mathcal{H}_2)$ can be written in the form [8]

$$\Gamma(\rho) = \sum_k V_k \rho V_k^*$$

where $\mathcal{V} = \{V_k\}_k$ is a collection of bounded operators $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and satisfy $\sum_k V_k^* V_k = \mathrm{I}$. This form is called operator-sum representation or Kraus decomposition. We call $\mathcal{V}$ a generator of Γ (see e.g. [5]).

Appendix C: Quantum binary channel

When the input system $\mathcal{H}_1$ and the output system $\mathcal{H}_2$ are both $\mathbf{C}^2$, a quantum channel is called a *quantum binary channel*. A density operator of the system is uniquely represented by a 2×2 Hermitian matrix of the form

$$\rho_\theta = \frac{1}{2}\begin{pmatrix} 1+\theta_3 & \theta_1 - i\theta_2 \\ \theta_1 + i\theta_2 & 1-\theta_3 \end{pmatrix}$$

where $\theta = (\theta_1, \theta_2, \theta_3)^t$ is called Stokes parameter and lies in the unit ball

$$\mathcal{V} = \{\theta \in \mathbf{R}^3 \,;\, \|\theta\|^2 = \theta_1^2 + \theta_2^2 + \theta_3^2 \le 1\}.$$

Then arbitrary quantum binary channel is represented as $\Gamma(\rho_\theta) = \rho_{A\theta+b}$ by a 3×3 real matrix A and a 3-dimensional real column vector b. For representing a completely positive map, they should satisfy the following condition [5]

$$\begin{pmatrix} \frac{1}{2}+p & x & r & w \\ \overline{x} & \frac{1}{2}-p & y & -r \\ \overline{r} & \overline{y} & \frac{1}{2}+q & z \\ \overline{w} & -\overline{r} & \overline{z} & \frac{1}{2}-q \end{pmatrix} \geq 0$$

when A and b are represented as

$$A = \begin{pmatrix} y_R + w_R & y_I + w_I & x_R - z_R \\ y_I - w_I & -y_R + w_R & -x_I + z_I \\ 2r_R & 2r_I & p-q \end{pmatrix}, \quad b = \begin{pmatrix} x_R + z_R \\ -x_I - z_I \\ p+q \end{pmatrix}.$$

(The subscripts R and I denote the real and imaginary parts, i.e. $x = x_R + ix_I$, etc.) We denote such a channel by $\Gamma = (A, b)$.

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Table 1: Examples of quantum binary channels (A, b)

	A	b
Γ_1	$\begin{pmatrix} 0.5 & 0 & 0 \\ 0 & 0.4 & 0 \\ 0 & 0 & 0.2 \end{pmatrix}$	$\begin{pmatrix} 0.2 \\ 0 \\ 0 \end{pmatrix}$
Γ_2	$\begin{pmatrix} 0.05 & -0.2 & 0.4 \\ -0.2 & -0.05 & -0.2 \\ 0.2 & 0 & -0.5 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 0.1 \end{pmatrix}$
Γ_3	$\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix} \cdot \begin{pmatrix} -0.45 & 0 & 0 \\ 0 & 0.6 & 0 \\ 0 & 0 & -0.6 \end{pmatrix} \cdot \begin{pmatrix} 0.8 & 0.6 & 0 \\ 0.6 & -0.8 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0.2 \\ -0.2 \\ 0.2 \end{pmatrix}$
Γ_4	$\begin{pmatrix} 0.1 & -0.3 & 0 \\ -0.3 & -0.1 & -0.2 \\ 0 & 0 & -0.05 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0.2 \\ 0.55 \end{pmatrix}$

Table 2: Examples of generators of quantum channels acting on $\mathcal{T}(\mathbf{C}^3)$ (We use the generators which include three operators and $V_3 = \sqrt{I - V_1^* V_1 - V_2^* V_2}$.)

	V_1	V_2
Γ_5	$\begin{pmatrix} 0.2 & 0.3 & 0.4 \\ 0 & 0.5i & 0 \\ 0.1i & 0.4i & 0.5i \end{pmatrix}$	$\begin{pmatrix} 0.1-0.3i & 0 & 0 \\ 0 & -0.3i & 0.1-0.2i \\ 0.3-0.3i & 0.2+0.1i & 0 \end{pmatrix}$
Γ_6	$\begin{pmatrix} 0.19 & 0.7 & -0.1+0.3i \\ 0.4i & 0.06 & -0.1+0.05i \\ 0.2 & 0.39 & 0.4-0.4i \end{pmatrix}$	$\begin{pmatrix} 0.3 & -0.1 & 0.1 \\ 0.2 & 0.3 & 0.02i \\ 0.1 & 0.2 & 0.1i \end{pmatrix}$

Table 3: Numerical values of the capacity of the quantum channels shown in Table 1 and 2 and their product channels

$\Gamma^{(1)}$	$\Gamma^{(2)}$	$C(\Gamma^{(1)})$	$C(\Gamma^{(2)})$	$C(\Gamma^{(1)}) + C(\Gamma^{(2)})$	$C(\Gamma^{(1)} \otimes \Gamma^{(2)})$
Γ_1	Γ_1	0.138166	0.138166	0.276311	0.276311
Γ_2	Γ_2	0.258679	0.258679	0.517358	0.517358
Γ_1	Γ_3	0.138166	0.243068	0.381233	0.381233
Γ_2	Γ_3	0.258679	0.243068	0.501747	0.501746
Γ_2	Γ_4	0.258679	0.0898225	0.348501	0.348501
Γ_5	Γ_5	0.677358	0.677358	1.354716	1.354716
Γ_6	Γ_6	0.829580	0.829580	1.659160	1.659160
Γ_5	Γ_6	0.677358	0.829580	1.506938	1.506938

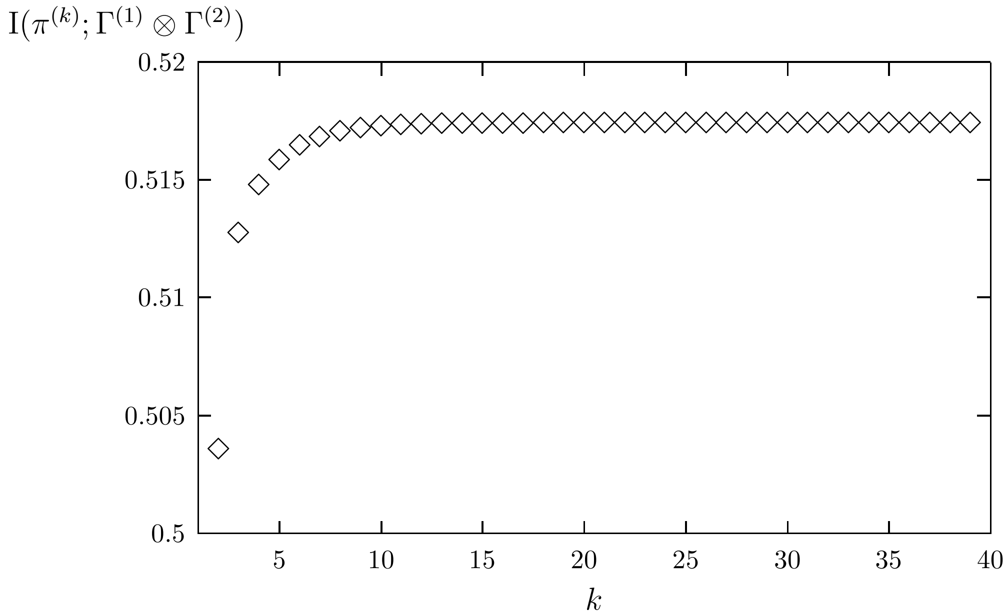

Figure 1: A change with iteration in quantum mutual information $\mathrm{I}(\pi^{(k)}; \Gamma^{(1)} \otimes \Gamma^{(2)})$.

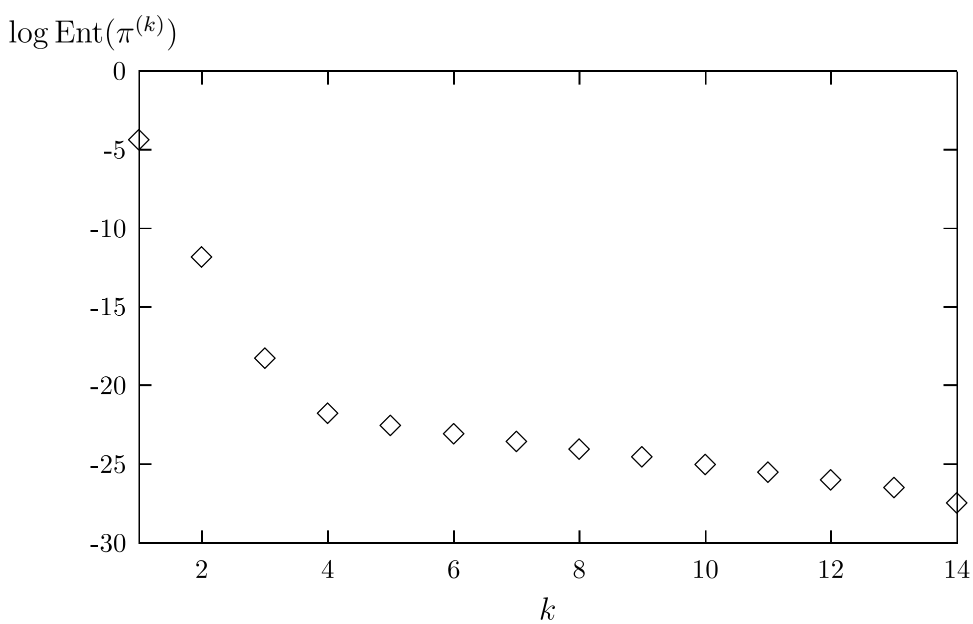

Figure 2: Semi-logarithmic plot of entanglement versus iteration number.